International Journal of Research in Engineering, Science and Management
Volume-3, Issue-5, May-2020
www.ijresm.com | ISSN (Online): 2581-5792

On Some Properties of Middle Cube Graphs and their Spectra

D. V. M. Ramani[1*], Mohammed Irshad Ali[2], Kota Ravi Manohar[3]

[1,2]*Associate Professor, Department of Basic Humanities and Sciences, Lords Institute of Engineering & Technology, Hyderabad, India*

[3]*Research Scholar, K. L. University, Vijayawada, India*

*Corresponding author: dvmramani@yahoo.in

***Abstract*: In this research paper we begin with study of family of n dimensional hyper cube graphs and establish some properties related to their distance, spectra, and multiplicities and associated eigen vectors and extend to bipartite double graphs[11]. In a more involved way since no complete characterization was available with experiential results in several inter connection networks on this spectrum our work will add an element to existing theory.**

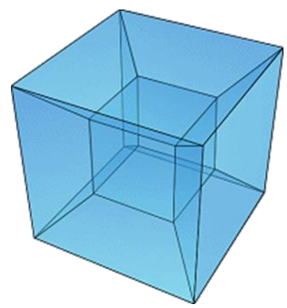
***Keywords*: Middle cube graphs, Distance-regular graph, Antipodal graph, Bipartite double graph, Extended bipartite double graph, Eigen values, Spectrum, Adjacency matrix.**

1. Introduction

An n dimensional hyper cube Q_n [24] also called n-cube is an "n" dimensional analogue of Square and a Cube. It is stopped up dense arched figure whose 1-skelton comprises of gatherings of inverse equal line fragments adjusted in every one of spaces measurements, opposite to one another and of same length. A factor is a hypercube of dimension zero. If one strikes this factor one-unit length, it will sweep out a line segment, which is the measure polytope of dimension one. If one strikes this line phase its size in a perpendicular course from itself; it sweeps out a two-dimensional square. If one strikes the rectangular one-unit size in the route perpendicular to the plain surface it lies on, it will generate a three-d cube. This can be generalized to any variety of dimensions. For example, if one strikes the dice one-unit size into the fourth dimension, it generates a four-dimensional measure polytope or tesseract.

The group of hypercubes is one of only a handful barely any normal polytopes that are spoken to in any number of measurements. The dual polytope of a hypercube is called a cross-polytope.

A hypercube of dimension n has $2n$ "sides" (a 1-dimensional line has 2 end points; a 2-dimensional square has 4 sides or edges; a 3-dimensional cube has 6 faces; a 4-dimensional tesseract has 8 cells). The number of vertices (points) of a hypercube is 2^n (a cube has 2^3 vertices, for instance).



The number of *m*-dimensional hyper cubes on the boundary of an *n*-cube is

$$2^{n-m}\binom{n}{m}$$

For example, the boundary of a 4-cube contains 8 cubes, 24 squares, 32 lines and 16 vertices.

A unit hyper cube is a hyper cube whose side has length 1 unit whose corners are

$$V_{2^{n+1}} \leftarrow \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}$$

2^n Points in R^n with every organize equivalent to 0 or 1 termed as measure polytope.

The correct number of edges of cube of dimension n is $n*2^{n-1}$ for example 7-cube has $7*2^6$ =448 edges.

A. Dimension of the cube

	1	2	3	4	5	6
No. of vertices	2	4	8	16	32	64
No. edges	1	4	12	32	80	192

Here we define adjacency matrix of n cube described in a constructive way.

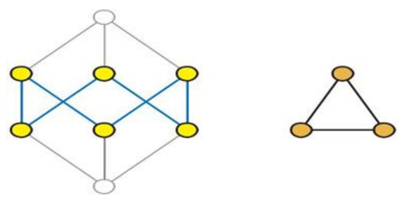
$$V_{2^{n+1}} \leftarrow \begin{pmatrix} V_{2^n} & I_{2^n} \\ I_{2^n} & V_{2^n} \end{pmatrix}$$

Since n Q_n is n regular bipartite graph of 2^n vertices characteristic vector of subsets of $[\mathrm{n}]=\{1,2,3,\ldots \mathrm{n}\}$ vertices of layer L_k corresponds to subsets of cardinality k.

If n is odd n=2k-1,the middle two layers L_k, L_{k-1} of Q_n with nc_k, nc_{k-1} vertices forms middle cube graph MQ_k by induction.

As MQ_k is bipartite double graph which is a sub graph of n-cube Q_n induced by vertices whose binary representations have either k-1 or k no. Of 1's is of k-regular as shown in figures below.

The middle cube graph MQ_2 as a subgraph of Q_3 or as the bipartite double graph of $O_2 = K_3$.



We start with spectral properties of bipartite double graphs [17], [18] and extend for study of eigen values of MQ_k .

B. Bipartite double graph

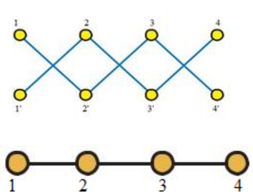
Let H= (V, E) be a graph of order n, with vertex set V = {1, 2…. n}. Its bipartite double graph $1+\lambda, -1-\lambda\hat{H}$ $\bar{H}$ = ($\bar{V}, \bar{E}$) is the graph with the projected vertex set,

$\bar{V}$ = {1,2… n.$1', 2', \ldots \mathrm{n}'$ }and adjacencies induced from the adjacencies in H as follows:

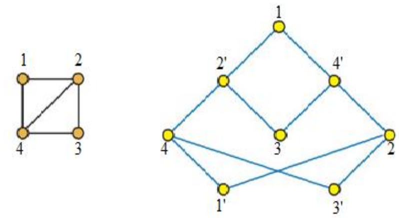
$$i \sim j \Longrightarrow \begin{cases} i \overset{E}{\sim} j' \\ j \overset{E}{\sim} i' \end{cases}$$

Thus, the edge set of $\bar{H}$ is $\bar{E}$ = $\{\mathrm{ij}' \mid \mathrm{ij} \in \mathrm{E}\}$. From the definition, it follows that $\bar{H}$

Is a bipartite graph [24.21] with stable subsets V_1= {1, 2…. n}, and V_2= $\{1', 2', \ldots \mathrm{n}'\}$. For example, if H is a bipartite graph, then its bipartite double graphs $\bar{H}$ consists of two non-connected copies of H.



Path p-4 and its bipartite Double Graph



Graph H has diameter 2 and $\bar{H}$ has diameter 3

If H is a δ-regular graph, then $\bar{H}$ also, if the degree sequence of the original graph H is,

δ = ($\delta_1, \delta_2, \delta_3 \ldots . \delta_n$), the degree sequence for its bipartite double graph is $\bar{\delta}$ =($\delta_1, \delta_2, \delta_3 \ldots . \delta_n, \delta_1, \delta_2, \delta_3 \ldots . \delta_n$)

The separation between vertices in the bipartite twofold diagram H can be given as far as the even and odd distances in H.

$$dist_{\bar{H}}(\mathrm{i}, \mathrm{j}) = \mathrm{dist}_H^+(\mathrm{i}, \mathrm{j})$$

$$dist_{\bar{H}}(\mathrm{i}, \mathrm{j}') = \mathrm{dist}_H^-(\mathrm{i}, \mathrm{j})$$

Involutive auto morphism with no fixed boundary lines, exchanging vertices i and i', the function from

$\bar{H}$ Onto H defined $i' \to i, i \to i$ is a 2-fold cover.

If $\hat{H}$ is extended bipartite double graph by adding edges (i,i')f or each $i \in V$ $\bar{H} \equiv \hat{H}$.

C. Notations

The cardinality of the graph G is n = {V} and its size is m = {E}. Name the vertices using natural numbers 1,2,…, n. If i is adjacent to j, that is, ij $\in$ E, we write i$\sim$ j or i$\overset{(E)}{\sim}$ j. The length between two ends is denoted by dist(I,j). We also use the concepts of even length and odd length between two ends, denoted by dist + and dist -, respectively. They are defined as the length of a shortest even, odd walk between the corresponding vertices. The set of vertices which are L-apart from vertex i, in view of the usual length, is $\Gamma_l(i) = \{j : dist(\mathrm{i}, \mathrm{j}) = l$, hence the degree of vertex is simply $\Gamma_l(i)$. The eccentricity of a vertex is ecc(i)= $\max_{1 \le X_{1 \le j \le n}} dist(\mathrm{i}, \mathrm{j})$ max1j_n dist(i; j) and the diameter of the graph is D =D(G) $\max_{1 \le X_{1 \le j \le n}} dist(\mathrm{i}, \mathrm{j})$ graph G`, G has the same vertex set as G and two vertices are close in G' if and just in the event that they are at unit distance in G. An antipodal diagram G is an associated chart of measurement D for which GD is a disjoint association of coteries. The collapsed diagram of G is the chart G whose vertices are the maximal clubs.

Let G = (V;E) be a graph with adjacency matrix A and λ-eigenvector v. Then, the charge of vertex i $\in$ V is the entry via v, and the equation $Av = \lambda v$.$\Rightarrow$eigen values of the bipartite double graph [11,16] $\bar{G}$ and the comprehensive bipartite double graph $\hat{G}$ as mappings of the eigen-values of a non-bipartite graph G.

We concentrate on some more outcomes which are less rudimentary yet significant on spectra multiplicities of related eigen vectors reached out to bipartite two fold charts.

2. Eigenvalues of the Graphs

Definition 1. *For a matrix* $A \in R^{m*n}$, *a number* λ *is an eigenvalue if for some vector* $x \neq 0$,

$$Ax = \lambda x.$$

The vector x is called an eigenvector corresponding to λ.
Some basic properties of eigenvalues are

- The eigenvalues are exactly the numbers λ that make the matrix $A-\lambda I$ singular, i.e. solutions of $\det(A-\lambda I) = 0$.
- All eigenvectors corresponding to λ form a subspace V_λ; the dimension of V_λ is called the multiplicity of λ.
- In general, eigenvalues can be complex numbers. However, if *A* is a symmetric matrix $(a_{ij} = a_{ji})$, then all eigenvalues are real, and moreover there is an orthogonal basis consisting of eigen vectors.
- The total of all eigenvalues, involving all multiplicities, is $\sum_{i=1}^{n} \lambda_i = Tr(A) = \sum_{i=1}^{n} a_{ij}$ the trace of A.
- The product of all eigenvalues, counting multiplicities, is $\prod_{i=1}^{n} \lambda_i = \det(A)$ the determinant of A.
- The figure of non-zero eigenvalues, including multiplicities, is the rank of *A*.

For graphs, we define eigenvalues as the eigenvalues of the *adjacency matrix*.

Definition 2. For a graph G, the adjacency matrix A(G) is defined as follows:

- $a_{ij} = 1 if\ (i, j) \in E(G)$
- $a_{ij} = 0 if i = j or (i, j) \notin E(G)$

since Tr (A(G)) = 0, follows,

The total of all eigen-values of a chart is always 0.

The (ordinary) spectrum of a diagram is the spectrum of its (0,1) adjacency matrix.

The graph on n vertices without edges (the n-coclique, $\overline{K}_n$) has zero adjacency

matrix, hence spectrum 0^n, where the exponent denotes the multiplicity

Complete bipartite graphs

The complete bipartite graph $K_{m,n}$ has spectrum $\pm\sqrt{mn}, 0^{m+n-2}$

More usually, every bipartite graph has a spectrum that is symmetric w.r.t.

the origin: if θ is eigenvalue, then also $-\theta$, with the same multiplicity.

The n-cube graph (called 2^n, or Q_n) is the chart by means of vertices the binary

vectors of length n, where two vectors are adjacent what time they be different in a single position. The 0-cube is K_1, the 1-cube is K_2, the 2-cube is C_4.

The spectrum of 2^n consists of the eigenvalues n − 2i with multiplicity $\binom{n}{i} (0 \le i \le n)$

The absolute bipartite graph $K_{m,n}$ has an adjacency matrix of rank 2, consequently we expect to have eigenvalue 0 of multiplicity *n*-2, and two non-trivial eigenvalues. These should be equal to $\pm\lambda$, because the sum of all eigenvalues is always 0.

We find λ by solving $Ax = \lambda x$. By symmetry, we guess that the eigenvector *x* should have *m*

Coordinates equal to α and *n* coordinates equal to β Then,

$$Ax = (m\beta, ..., m\beta, n\alpha, ...n\alpha)$$

This should be a multiple of $x = (\alpha, ..., \alpha, \beta, ..., \beta)$. Therefore, we get $m\beta = \lambda\alpha$ and $n\alpha = \lambda\beta$ i.e. and $mn\beta = \lambda^2\beta$ and $\lambda = \sqrt{mn}$

A graph Γ is called bipartite when its vertex set can be partitioned into two disjoint parts $X_1 X_2$ such that all edges of Γ meet both X_1 and X_2. The adjacency

matrix of a bipartite graph has the form $A = \begin{Bmatrix} 0 & B \\ B^T & 0 \end{Bmatrix}$. It follow that the spectrum of a bipartite graph is symmetric w.r.t. 0: if $\begin{bmatrix} u \\ v \end{bmatrix}$ is an eigenvector with eigenvalue θ, then $\begin{bmatrix} u \\ -v \end{bmatrix}$ is an eigenvector with eigenvalue −θ.

For the ranks one has rkA = 2 rk B. If n_i = |Xi| (i = 1, 2) and n1 ≥ n2, then rkA ≤ 2n2, so that Γ has eigenvalue 0 with multiplicity at least n1 − n2.

One cannot, in general, recognize bipartiteness from the Laplace or signless Laplace spectrum. Eg., $K_{1,3}$ and $K_1 + K_3$ have the same signless Laplace spectrum and only the former is bipartite.

However, by Proposition below, a graph is bipartite precisely when its
Laplace spectrum and signless Laplace spectrum coincide.

A. Elementary Graphs associated Eigen values

label	picture	A	L	Q	R
0.1					
1.1	•	0	0	0	0
2.1		1,-1	0.2	2.0	-1,1
2.2		0,0	0,0	0,0	-1,1
3.1		2,-1,-1	0,3,3	4,1,1	-2,1,1
3.2		$\sqrt{2},0,-\sqrt{2}$	0,1,3	3,1,0	-1,-1,2
3.3		1,0,-1	0,0,2	2,0,0	-2,1,1
3.4		0,0,0	0,0,0	0,0,0	-1,-1,2

B. Characteristic polynomial

Let Γ be a directed graph on n vertices. For any directed sub-graph C of Γ that is a union of directed cycles, let c(C) be its figure of cycles. Then the trait polynomial
pA(t) =det(tI −A) of Γ can be expanded as $\sum C_i t^{n-i}$ where $C_i = \sum_C (-1)^{c(C)}$ with C association over all usual heading for sub graphs with in- and outdegree 1 on i vertices.
(Indeed, this is just a reformulation of the definition of the determinant as,
$\det M = \sum_{\sigma} \operatorname{sgn}(\sigma) M_{1\sigma(1)} \ldots M_{n\sigma(n)}$ Note that when the variation σ with n−i fixed points is written as a product of non-identity cycles, its sign is $(-1)^e$.where e is the number of even cycles in this product. Since the number of odd non-identity cycles is congruent to i(mod 2), we have $sgn(\sigma) = (-1)^{i+c(\sigma)})$
For example, the triangle has $c_0 = 1, c_3 = -1$. Directed edges that do not occur in directed cycles do not influence the (ordinary) spectrum.
The same description of $p_A(t)$ holds for undirected graphs (with each edge viewed as a pair of opposite directed edges).
Since $\frac{d}{dt}\det(tI - A) = \sum_x \det(tI - A_x)$ where A_x is the sub-matrix of A obtained by deleting row and column x, it follows that $p'_A(t)$ is the sum of the characteristic polynomials of all single-vertex-deleted sub-graphs of Γ.
The spectrum of the complete bipartite graph $K_{m,n} is \pm\sqrt{mn},\ 0^{m+n-2}$. The Laplace spectrum is $0^1,\ m^{n-1},\ n^{m-1}, (m+n)^1$

The main eigenvalue of a graph is also recognized as its spectral radius or index. The basic in sequence concerning the main eigenvalue of a (possibly directed) graph is provide by Perron-Frobenius theory as follows.

C. Proposition

Each graph Γ has a real eigenvalue θ_0 with positive real corresponding eigen-vector, and such that for each eigenvalue θ we have |θ| ≤ θ_0.
The value θ_0 (Γ) remains same when vertices or edges are removed from Γ.
Let Γ is strongly connected. Then
(i) θ_0 has multiplicity 1.
(ii) If Γ is primitive (strongly connected, and such that no cycles have a
length equal to multiple of some integer
d > 1), then |θ| < θ_0 for all
eigen-values θ different from θ_0.
(iii) The value θ_0 (Γ) reduces when vertices or edges are removed from Γ
Now let Γ be undirected. By Perron-Frobenius theory and interlacing we
find an upper and lower bound for the maximum characteristic value of a connected graph.
(Note that A is irreducible if and only if Γ is connected.)
Out of all the connected graphs Γ, with non-primitive A are precisely the bipartite
diagram (with period 2) is explained in the following proposition.

D. Proposition

(i) A graph Γ is bipartite if and only if for each eigenvalue θ of Γ, also −θ is an eigen-value, with the same multiplicity.
(ii) If Γ is connected with largest eigen-value θ_1, then Γ is bipartite if and only
if $-\theta_1$ is an eigen-value of Γ.
Proof. For connected graphs all is clear from the Perron-Frobenius theorem.
That gives (ii) and (by taking unions) the 'only if' part of (i). For the 'if' part
of (i), let θ_1 be the spectral radius of Γ. Then some connected component of Γ
Has eigenvalues θ_1 and $-\theta_1$, and hence is bipartite. Removing its contribution
to the spectrum of Γ, we see by acceptance on the quantity of parts that all Parts are bipartite.

We build up some more hypotheses stretched out on spectra and multiplicities and related eigen values which are reached out to bipartite twofold charts.

Theorem: Let F be a field, let R be a commutative sub ring to F^{n*n}, the set of all n *n
Matrices over F. Let M ∈ R^{m*m} , then

$$\det_F (M) = \det_F(\det_R(M))$$

$$\therefore \det_F (M) = \det_F(AD - BC).$$

for a bipartite twice chart feature polynomial. [13]

We show the next theorems presentation geometric multiplicities of eigen value λ of H $\Rightarrow$ geometric multiplicities of eigen values λ and $-\lambda$ of $\bar{H}$ $1+\lambda, -1-\lambda$ of $\hat{H}$

Theorem: Let H be a graph on n vertices, with the adjacency matrix A and characteristic,

$$\bar{u} = u_i^+ v_i (1+\lambda)$$

$$(u^+)_{i'} = \sum_{j \sim i'}^{E} u \overset{+}{j} = \sum_{j \sim i}^{E} v_j = \lambda v_i = \lambda u_i^+$$

polynomial $\varnothing_H$ (x). Then, the characteristic polynomials of $\bar{H}$ and $\hat{H}$ are, respectively,

$$\varnothing_{\bar{H}}(x) = (-1)^n \varnothing_H(x) \varnothing_H(-x),$$

$$\varnothing_{\hat{H}}(x) = (-1)^n \varnothing_H(x-1) \varnothing_H(-x-1).$$

Adjacency matrices are, in that order,

$$\bar{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \text{and } \hat{A} = \begin{pmatrix} O & A+I \\ A+I & O \end{pmatrix}.$$

With the above corollary

$$\varnothing_{\bar{H}}(x) = \det(xI_{2n} - \bar{A}) = \det\begin{pmatrix} xI_n & -A \\ -A & xI_n \end{pmatrix} = \det(x^2 I_n - A^2)$$

$$= \det(xI_n - A)\det(xI_n + A) = (..1)^n \varnothing_H(x) \varnothing_H(-x);$$

Whereas the characteristic polynomial of $\hat{H}$ is

$$\varnothing_{\hat{H}}(x) = \det(xI_{2n} - \hat{A}) = \det\begin{pmatrix} xI_n & -A-I_n \\ -A-I_n & xI_n \end{pmatrix}$$

$$= \det(x^2 I_n - (A+I_n)^2) = \det(xI_n - (A+I_n))\det(xI_n + (A+I_n))$$

$$= \det((x-1)I_n - A)(-1)^n \det(-(x+1)I_n - A)$$

$$= (-1)^n \varnothing_H(x-1) \varnothing_H(-x-1).$$

Theorem: Let H be a graph and v a λ-eigenvector H. Let us consider the vector u+ with Components $u_i^+ = u_{i'}^+ = v_i$, u- with components $u_i^- = v_i$ and $u_{i'}^- = -v_i$ for $1 \le i, i' \le n$

Then,

u^+ λ-eigenvector $\bar{H}$ and $(1+\lambda)$ eigenvector $\hat{H}$

$\bar{u}$ - λ-eigenvector $\bar{H}$ and $(-1-\lambda)$ eigenvector $\hat{H}$

If the vertex i, $1 \le i \le n$, all its nearby vertices are of type j', with i (E) $\sim$ j. Then

$$(Au^+)_i = \sum_{j \sim i'}^{E} u \overset{+}{j} = \sum_{j \sim i}^{E} v_j = \lambda v_i = \lambda u_i^+$$

If the vertex I', $1 \le i \le n$, all its neighboring vertices are of type j, with i (E) $\sim$ j.

Then

$$(Au^+)_{i'} = \sum_{j \sim i'}^{E} u \overset{+}{j} = \sum_{j \sim i'}^{E} v_j = \lambda v_i = \lambda u_i^+$$

By a comparable reasoning with u^-, we obtain

$$(Au^-)_i = \sum_{j \sim i'}^{E} u \overset{+}{j}' = -\sum_{j \sim i}^{E} v_j = -\lambda u_{i-} \text{ and}$$

$$(Au^-)_{i'} = \sum_{j \sim i'}^{E} u \overset{-}{j} = \sum_{j \sim i}^{E} v_j = -\lambda u_i'$$

$m(\lambda_0) = m(\lambda_5) = m(\theta_0^{\pm}) = 1,$

$m(\lambda_1) = m(\lambda_4) = m(\theta_1^{\pm}) = 4,$

$m(\lambda_2) = m(\lambda_3) = m(\theta_2^{\pm}) = 5,$

$\therefore$ u^- is -λ-eigenvector of bipartite double graph $\bar{H}$.

Also 1+λ,-1-λ are eigen values for u^+, u^- eigen vectors of $\hat{H}$

From the beyond information realizing an isomorphism [8, 2] defined by

$$f : V[\tilde{O}_k] \to V[MQ_k]$$
$$\mathbf{u} \mapsto \mathbf{u}$$
$$\mathbf{u'} \mapsto \bar{\mathbf{u}}$$

is a surjection by the definition of bipartite double graph, if u and v' Are two vertices of $\tilde{O}_k$.

The center cube graph $[MQ_k]$ with D=2k-1 by above corollary is isomorphic to $\tilde{O}_k$.

We authenticate spectrum of Q_{2k-1} hold all eigen values of $[MQ_k]$,

$\theta_i^+ = (-1)^i$(k-i) and = $\theta_i^- = -\theta_i^+$ for $0 \le i \le k-1$

With multiplicities $m(\theta_i^+) = m(\theta_i^-) = \frac{k-1}{k}\binom{2k}{i}$

3. Conclusion

In Verification of the above results,

$$spMQ_3 = \{\pm 2, \pm 1^2\}$$
$$spMQ_5 = \{\pm 3, \pm 2^4, \pm 1^5\}$$
$$spMQ_7 = \{\pm 4, \pm 3^6, \pm 2^{14}, \pm 1^{14}\}$$
$$spMQ_9 = \{\pm 5, \pm 4^8, \pm 3^{27}, \pm 2^{48}, \pm 1^{42}\}$$

For uppermost degree Distance polynomials of $[MQ_k]$

p5 (3) = p5 (1) = p5 (-1) = 1 and p5 (2) =p5 (-1) = p5 (-3) = -1.

Then,

$$m(\lambda_0) = m(\lambda_5) = m(\theta_0^{\pm}) = 1,$$
$$m(\lambda_1) = m(\lambda_4) = m(\theta_1^{\pm}) = 4,$$
$$m(\lambda_2) = m(\lambda_3) = m(\theta_2^{\pm}) = 5,$$

References

[1] A. E. Brouwer, A. M. Cohen, and A. Neumaier, Distance-Regular Graphs, Springer, Berlin (1989).
[2] B. Mohar, Eigenvalues, diameter and mean distance in graphs, Graphs Combin. Theory Ser.B 68 (1996), 179–205.
[3] C. D. Godsil, Algebraic Combinatorics, Chapman and Hall, London/New York (1993).
[4] C. D. Godsil, More odd graph theory, Discrete Math. 32 (1980), 205–207.
[5] C. D. Savage, and I. Shields, A Hamilton path heuristic with applications to the middle two levels problem, Congr. Numer. 140 (1999), 161–178.
[6] C. Delorme, and P. Sol´e, Diameter, covering index, covering radius and eigenvalues, European J. Combin. 12 (1991), 95–108. Distance-regular graphs, J. Graph Theor. 27 (1998), 123–140.
[7] E. R. van Dam, and W. H. Haemmers, Eigenvalues and the diameter of graphs, Linear and Multilinear Algebra 39 (1995), 33–44.
[8] F. Harary, J. P. Hayes, and H. J. Wu, A survey of the theory of hypercube graphs, Comp.Math. Appl. 15 (1988), no. 4, 277–289.
[9] F. R. K. Chung, Diameter and eigenvalues, J. Amer. Math. Soc. 2 (1989), 187–196.
[10] Havel, Semipaths in directed cubes, in M. Fiedler (Ed.), Graphs and other Combinatorial Topics, Teunebner–Texte Math., Teubner, Leipzig (1983).
[11] I. Bond, and C. Delorme, New large bipartite graphs with given degree and diameter, A. Combin. 25C (1988), 123–132.
[12] J. Hoffman, On the polynomial of a graph, Amer. Math. Monthly 70 (1963), 30–36.
[13] J. R. Silvester, Determinants of block matrices, Maths Gazette 84 (2000), 460–467.
[14] J. Robert Johnson, Long cycles in the middle two layers of the discrete cube, J. Combin. Theory Ser. A 105 (2004), 255–271.
[15] K. Qiu, and S. K. Das, Interconnexion Networks and Their Eigenvalues, in Proc. of 2002 International Sumposiym on Parallel Architectures, Algorithms and Networks, ISPAN'02, pp. 163–168.
[16] M. A. Fiol, Algebraic characterizations of distance-regular graphs, Discrete Math. 246(2002), 111–129.
[17] M. A. Fiol, E. Garriga, and J. L. A. Yebra, Boundary graphs: The limit case of a spectral property, Discrete Math. 226 (2001), 155–173.
[18] M. A. Fiol, E. Garriga, and J. L. A. Yebra, Boundary graphs: The limit case of a spectral property (II), Discrete Math. 182 (1998), 101–111.
[19] M. A. Fiol, E. Garriga, and J. L. A. Yebra, From regular boundary graphs to antipodal
[20] M. A. Fiol, E. Garriga, and J. L. A. Yebra, On a class of polynomials and its relation with the spectra and diameter of graphs, J. Combin. Theory Ser. B 67 (1996), 48–61.
[21] M. A. Fiol, E. Garriga, and J. L. A. Yebra, on twisted odd graphs, Combin. Probab. Comput.9 (2000), 227–240.
[22] M.A. Fiol, and M. Mitjana, The spectra of some families of digraphs, Linear Algebra Appl.423 (2007), no. 1, 109–118.
[23] N. Alon and V. Milman, _1, Isoperimetric inequalities for graphs and super-concentrators, J.Combin. Theory Ser. B 38 (1985), 73–88.
[24] N. Biggs, Algebraic Graph Theory, Cambridge University Press, Cambridge (1974), secondedition (1993).
[25] N. Biggs, An edge coloring problem, Amer. Math. Montly 79 (1972), 1018–1020.
[26] N. Biggs, Some odd graph theory, Ann. New York Acad. Sci. 319 (1979), 71–81.
[27] T. Balaban, D. Farcussiu, and R. Banica, Graphs of multiple 1; 2-shifts in carbonium ions and related systems, Rev. Roum. Chim. 11 (1966), 1205–1227.
[28] J.Robert Johnson, Long cycles in the middle twp layers of the diserete cube, J. Combin. heors Ser. A 105 (2004). 255-271.
[29] C. D. Savage. and 1. Shields, A Hamilton path heuristic with applications to the midkdle two levels problem, Congr, Numer, 140 (1999). 161–178.
[30] J. R. Silvester, Determinants of block matrices, Maths Gazette 84 (2000), 460-467.